3+1 spinfoam model of quantum gravity with spacelike and timelike components

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We present a spinfoam formulation of Lorentzian quantum General Relativity. The theory is based on a simple generalization of an Euclidean model defined in terms of a field theory over a group. The model is an extension of a recently introduced Lorentzian model, in which both timelike and spacelike components are included. The spinfoams in the model, corresponding to quantized 4-geometries, carry a natural non-perturbative local causal structure induced by the geometry of the algebra of the internal gauge ($sl(2, C)$). Amplitudes can be expressed as integrals over the spacelike unit-vectors hyperboloid in Minkowski space, or the imaginary Lobachevskian space.

I. INTRODUCTION

Most of the work in non-perturbative quantum gravity is restricted to the unphysical Euclidean sector, deferring the construction of the physical Lorentzian theory. A step towards the Lorentzian theory has been recently taken in Refs. [1,2], with the definition of spinfoam models based on $SL(2, C)$ representation theory. However, these models includes only (simple) representations in the continuous series, and not the ones in the discrete series. Since the signature of the surfaces in the spinfoam is determined by the sign of the Casimir, which is opposite in the two series, all surfaces of the model turn out to have the same signature. In a more realistic model, on the other hand, we expect spacelike as well as timelike surfaces to appear, and thus all (simple) representation to contribute. In this paper, we introduce a new model, which includes both kinds of representations. As a consequence, the model incorporates a combinatorial non-perturbative notion of local causality associated to quantum spacetime. Surfaces in a given spinfoam can be classified as timelike or spacelike, according to the kind of simple representations by which they are colored.

We recall that spinfoam models provide a framework for background independent diffeomorphism-invariant quantum field theory and quantum gravity in particular [3–9]. They provide a rigorous implementation of the Wheeler-Misner-Hawking [10,11] sum over geometries formulation of quantum gravity. The 4-geometries summed over are represented by spinfoams. These are defined as colored 2-complexes. A 2-complex J is a (combinatorial) set of elements called "vertices" v, "edges" e and "faces" f, and a boundary relation among these. A spinfoam is a 2-complex plus a "coloring" N, that is an assignment of an irreducible representation N_f of a given group G to each face f and of an intertwiner i_e to each edge e. The model is defined by the partition function

$$Z = \sum_J \mathcal{N}(J) \sum_N \prod_{f\in J} A_f(N_f) \prod_{e\in J} A_e(N_e) \prod_{v\in J} A_v(N_v), \tag{1}$$

where A_f, A_e and A_v correspond to the amplitude associated to faces, edges, and vertices respectively (they are given functions of the corresponding colors). $\mathcal{N}(J)$ is a normalization factor for each 2-complex.

Spinfoam models can be obtained as the perturbative expansion of a field theory over a group manifold [17]. In this language, spinfoams appear as Feynman diagrams of a scalar field theory over a group. More precisely equation (1) can be obtained by the perturbative expansion of

$$Z = \int \mathcal{D}\phi \; e^{-S[\phi]} \tag{2}$$

in momentum space. In particular, the topological models of BF theory can be described in this framework [18–21]. The field theory approach has several advantages. It implements automatically the sum over 2-complexes J, and in particular, fixes the $\mathcal{N}(J)$ value in (1). This sum restores full general covariance of a theory with local degrees of freedom such as general relativity (GR) [5,16]. Other advantages of the field theory formulation are related to the possibility of performing formal manipulations directly on configuration space of the field theory, which turn out to be much simpler than working directly with the state sum model of equation (1) (i.e., momentum space). Examples of this are the proof of topological invariance for the BF models due to Ooguri [19], and the definition of the Lorentzian model mentioned above. In this last case, the non-compactness of $SL(2, C)$ introduces additional complications that are easier to deal with in the field theory formulation.

Spinfoam models related to gravity have been obtained as modifications of topological quantum field theories by implementing the constraint that reduce BF theory to GR [9,12–14]. Essentially this constraint amounts to restrict to simple representations only. The resulting models correspond to a diffeomorpism invariant lattice-like quantization of Plebanski's formulation of GR [15]. In the field theory framework, the constraints reducing BF theory to GR are naturally implemented by imposing a symmetry requirement on the field action defining the BF model [16]. In the Euclidean models, the field is defined over $SO(4)$, and the BF to GR constraint is imposed as a symmetry under an $SO(3)$ subgroup. Amplitudes turn out to be expressed as integrals over the homogeneous space $S_3 = SO(4)/SO(3)$ and the theory is controlled by the harmonic analysis on S_3, which contains the simple representations of $SO(4)$. In the Lorentzian models introduced in Refs. [1,2], the field is defined over $SL(2, C)$, and the BF to GR constraint is imposed as a symmetry under an $SU(2)$ subgroup. Amplitudes turn out to be expressed as integrals over the homogeneous space $SL(2, C)/SU(2)$, which is the (real) Lobachevskian space: the upper hyperboloid of unit norm timelike vectors in in Minkowski space. The theory is controlled by the harmonic analysis on this space, which contains the continuous simple representations of $SL(2, C)$ only.

In the model we introduce here, the field is still defined over $SL(2, C)$, but the BF to GR constraint is imposed as a symmetry under an $SU(1, 1) \times Z_2$ subgroup. Amplitudes turn out to be expressed as integrals over the homogeneous space $SL(2, C)/(SU(1, 1) \times Z_2)$, which is the *imaginary* Lobachevskian space: the hyperboloid of unit norm spacelike vectors in in Minkowski space, with opposite points identified. The theory is controlled by the harmonic analysis on this space, which contains simple representations of $SL(2, C)$ in the continuous as well as in the discrete series. Thus, we obtain a model that implements the BF to GR constraints and in which the quantum four geometries summed over have spacelike as well as timelike surfaces.

The paper is organized as follows. In the next section we describe the general setting in which the Lorentzian spinfoam model of reference [2] was defined. The new model is defined in the same framework in terms of a simple modification of the one in [2]. Both variants are presented in the corresponding subsections. We obtain the edge and vertex amplitudes of the models as integrals over the Lobachevskian spaces of a kernel $K^{\pm}$. In section III we compute these kernels for the two models. In the appendix we present a compendium of known results on harmonic analysis and representation theory of $SL(2, C)$ on which our construction is based.

II. $SL(2, C)$ STATE SUM MODELS OF LORENTZIAN QG

In this section we briefly discuss the general framework of spinfoam models defined as a field theory over a group. In particular we review the derivation of the Lorentzian spinfoam model of [2]. The implementation of the simplicity constraints is encoded the symmetries of the interaction. Those symmetries are implemented in the field action by means of group averaging techniques. The new model differs from its previous relative in the kind of symmetries required (different implementation of the constraints). For that reason we start the section with the simultaneous treatment of both theories since the formal manipulation necessary to derive the model are the same in each one of them. In the two subsection that follow this general derivation we explicitly state the results for both models.

We start with a field $\phi(g_1, g_2, g_3, g_4)$ over $SL(2, C) \times SL(2, C) \times SL(2, C) \times SL(2, C)$. We assume the field has compact support and is symmetric under arbitrary permutations of its arguments. [1] We define the projectors P_γ and $P_u^{(\pm)}$ as

$$P_g\phi(g_i) \equiv \int d\gamma\ \phi(g_i\gamma), \tag{3}$$

and

$$P_u^{(\pm)}\phi(g_i) \equiv \int_{U^{(\pm)}} du_i^{(\pm)}\ \phi(g_i u_i), \tag{4}$$

with $\gamma \in SL(2, C)$, and $u_i \in U^{(\pm)} \subset SL(2, C)$, where $U^{(\pm)}$ correspond the $SL(2, C)$ subgroups defined as follows. Think of the vector representation of $SL(2, C)$, i.e., four dimensional irreducible representation defined on $\Re^4$. The $SL(2, C)$ action defines a Minkowski metric in $\Re^4$. Let it be of signature $(+, -, -, -)$. Consider the time-like line $(\lambda, 0, 0, 0)$ and the space-like line $(0, 0, 0, \lambda)$ for $\lambda \in \Re$, and associate to them the one parameter family of 2×2 matrices $\ell^+ = \lambda\sigma_0$ and $\ell^- = \lambda\sigma_3$ respectively (σ_μ being the identity matrix for $\mu = 0$ and the Pauli matrices for

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[1] This symmetry guarantees arbitrary 2-complexes J to appear in the Feynman expansion [16].

$\mu = 1, 2, 3$). We define the subgroup $U^{(\pm)} \subset SL(2, C)$ as the subgroup leaving invariant $\ell^{\pm}$ respectively under the action $\ell \to u \ell u^{\dagger}$. Clearly $U^{(+)}$ is isomorphic to $SU(2)$. $U^{(-)}$, on the other hand, is isomorphic to $SU(1, 1) \times Z_2$ [2]. Finally, $d\gamma$ and $du^{(\pm)}$ denote the corresponding invariant measures.

We define the following two actions which give rise to the spinfoam models considered in the paper

$$S^{(\pm)}[\phi] = \int dg_i \, [P_\gamma \phi(g_i)]^2 + \frac{1}{5!} \int dg_i \, [P_\gamma P_u^{(\pm)} \phi(g_i)]^5 \, , \tag{5}$$

where $\gamma_i \in SL(2, C)$, $\phi(g_i)$ denotes $\phi(g_1, g_2, g_3, g_4)$, and the fifth power in the interaction term is notation for

$$[\phi(g_i)]^5 := \phi(g_1, g_2, g_3, g_4) \; \phi(g_4, g_5, g_6, g_7) \; \phi(g_7, g_3, g_8, g_9) \; \phi(g_9, g_6, g_2, g_{10}) \; \phi(g_{10}, g_8, g_5, g_1). \tag{6}$$

The γ integration projects the field into the space of gauge invariant fields, namely, those such that $\phi(g_i) = \phi(g_i \mu)$ for $\mu \in SL(2, C)$.[3] We continue the construction of the spinfoam models corresponding to $S^{(+)}[\phi]$, and $S^{-}[\phi]$ in general and we particularize to each case in the following two subsections. The vertex and propagator of the theories are simply given by a set of delta functions on the group, as illustrated in [14], to which we refer for details. Feynman diagrams correspond to arbitrary 2-complex J with 4-valent edges (bounding four faces), and 5-valent vertices (bounding five edges). Once the configuration variables g_i are integrated over, the Feynman amplitudes reduce to integrals over the group variables γ and u in the projectors in (5). These end up combined as arguments of one delta functions per face [14]. That is, a straightforward computation yields

$$A^{(\pm)}(J) = \int_{U^{(\pm)}} du^{(\pm)} d\gamma \prod_e \prod_f \delta(\gamma^{(1)}_{e_1} u^{(\pm)}_{1f} \gamma^{(2)}_{e_1} u'^{(\pm)}_{1f} \gamma^{(3)}_{e_1} \dots \gamma^{(1)}_{e_N} u^{(\pm)}_{Nf} \gamma^{(2)}_{e_N} u'^{(\pm)}_{Nf} \gamma^{(3)}_{e_N}), \tag{7}$$

where e and f denote the set of edges and faces of the corresponding 2-complex J. In this equation, $\gamma^{(1)}_e$, and $\gamma^{(3)}_e$ come from the group integration in the projectors P_γ in the two vertices bounding the edge e. $\gamma^{(2)}_e$ comes from the projector P_γ in the propagator defining the edge e. Finally, $u^{(\pm)}_{1f}$ and $u'^{(\pm)}_{1f}$ are the integration variables in the projector P_h in the two vertices. Notice that each u integration variable appears only once in the integrand, while each γ integration variable appears in four different delta's (each edge bounds four faces). The index N denotes the number of edges of the corresponding face. Now we use equation (A26) to expand the delta functions in terms of irreducible representations of $SL(2, C)$. Only the representations (n, ρ) in the principal series contribute to this expansion. We obtain

$$A^{(\pm)}(J) = \sum_n \int_{\rho_f} d\rho \prod_f (\rho_f^2 + n_f^2) \int \prod_e d\gamma du^{(\pm)} \; \mathrm{Tr}\left[\bar{D}^{n_f \rho_f}(\gamma^{(1)}_{e_1} u^{(\pm)}_{1f} \gamma^{(2)}_{e_1} u'^{(\pm)}_{1f} \gamma^{(3)}_{e_1} \dots \gamma^{(1)}_{e_N} u^{(\pm)}_{Nf} \gamma^{(2)}_{e_N} u'^{(\pm)}_{Nf} \gamma^{(3)}_{e_N})\right]. \tag{8}$$

Next, we rewrite this equation in terms of the matrix elements $\bar{D}^{n\rho}_{j_1 q_1 j_2 q_2}(\gamma)$ of the representation (n, ρ) in the canonical basis, defined in the appendix. The trace becomes

$$\mathrm{Tr}\left[\bar{D}^{n_f \rho_f}(\gamma^{(1)}_{e_1} u^{(\pm)}_{1f} \gamma^{(2)}_{e_1} u'^{(\pm)}_{1f} \gamma^{(3)}_{e_1} \dots \gamma^{(1)}_{e_N} u^{(\pm)}_{Nf} \gamma^{(2)}_{e_N} u'^{(\pm)}_{Nf} \gamma^{(3)}_{e_N})\right] = \bar{D}^{n_f \rho_f}_{j_1 q_1 j_2 q_2}(\gamma^{(1)}_{e_1}) \bar{D}^{n_f \rho_f}_{j_2 q_2 j_3 q_3}(u^{(\pm)}_{1f}) \bar{D}^{n_f \rho_f}_{j_3 q_3 j_4 q_4}(\gamma^{(2)}_{e_1}) \dots \bar{D}^{n_f \rho_f}_{j.q.j_1 q_1}(\gamma^{(3)}_{e_N}). \tag{9}$$

Repeated indices are summed, and the range of the j_n and q_n indices is specified in the appendix. The integration of $\bar{D}^{n_f \rho_f}(u^{(\pm)}_{1f})_{j_2 q_2 j_3 q_3}$ over $u^{(\pm)}_{1f}$ is zero if in the corresponding representation there are no invariant vectors under the action of $U^{(\pm)}$.[4] It can be written as

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[2] The elements of Z_2 can be realized as the 2×2 matrices $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$ respectively.

[3] Because of this gauge invariance, the action (5) is proportional to the trivial diverging factor $\int d\gamma$. This divergence could be fixed easily, for instance by gauge fixing and just dropping one of the group integrations. For the clarity of the presentation, however, we have preferred to keep gauge invariance manifest, use the action formally to generate the Feynman expansion, and drop the redundant group integration whenever needed.

[4] This projection implements the constraint that reduces BF theory to GR. Indeed, the generators of $SL(2, C)$ are identified with the classical two-form field B of BF theory. The generators of the simple representations satisfy precisely the BF to GR constraint. Namely B has the appropriate $e \wedge e$ form [9,5].

$$\int_{U^{(\pm)}} du^{(\pm)} \bar{D}^{n\rho}(u^{(\pm)})_{j_1 q_1 j_2 q_2} = \mathcal{W}^{(\pm)n\rho}_{j_1 q_1} \bar{\mathcal{W}}^{(\pm)n\rho}_{j_2 q_2}, \tag{10}$$

where $\mathcal{W}^{(\pm)n\rho}_{jq}$ is the invariant vector under $U^{(\pm)}$ in the representation (n, ρ). According to (10), the magnitude of $\mathcal{W}^{(\pm)n\rho}_{jq}$ are given by the volume of $U^{(\pm)}$; as a consequence, $\mathcal{W}^{(-)n\rho}_{jq}$ are not normalizable. [5] Equation (10) defines

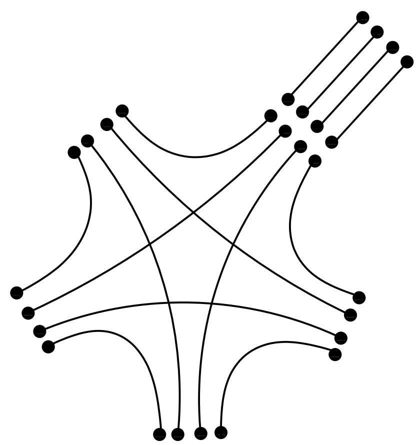

FIG. 1. Structure of the propagator and interaction. The black dots represent the projections $\mathcal{W}^{(\pm)}$ produced by the $U^{(\pm)}$ averaging in equation (7).

a projection operator only with the + sign, while in the other case we will have to take care of the infinite volume factor every time we square (10). This fact does not cause any problems in our deduction. This is addressed in section (III).

One of the two $\mathcal{W}^{(\pm)n\rho}$ in (10) appears always contracted with the indices of the $\bar{D}(\gamma)$ associated to a vertex; while the other is contracted with a propagator. We observe that the representation matrices associated to propagators ($\gamma^{(2)}_e$) appear in four faces in (8). The ones associated to vertices appear also four times but combined in the ten corresponding faces converging at a vertex. Consequently, they can be paired according to the rule $\bar{D}^{n\rho}_{jqkl}(\gamma_{e_i})\bar{D}^{n\rho}_{klst}(\gamma_{e_j}) = \bar{D}^{n\rho}(\gamma_{e_i}\gamma_{e_j})_{jqst}$. In Fig. (1) we represent the structure described above. A continuous line represents a representation matrix, while a dark dot a contraction with an invariant vector $\mathcal{W}^{(\pm)n\rho}$. Taking all this into account, and denoting $\bar{D}^{n\rho}_{\mathcal{W}^{(\pm)}\mathcal{W}^{(\pm)}}(\gamma)$ the matrix element $\bar{\mathcal{W}}^{(+)n,\rho}_{j_1 q_1} \bar{D}^{n,\rho}(\gamma)_{j_1 q_1 j_2 q_2} \mathcal{W}^{(+)n,\rho}_{j_2 q_2}$, we obtain

$$A^{(\pm)}(J) = \sum_{n_f} \int_{\rho_f} \prod_f (\rho_f^2 + n_f^2) \quad \prod_e A^{(\pm)}_e(\rho_{e_1}, \dots \rho_{e_4}; n_{e_1}, \dots n_{e_4}) \prod_v A^{(\pm)}_v(\rho_{v_1}, \dots \rho_{v_{10}}; n_{v_1}, \dots n_{v_{10}}), \tag{14}$$

where $A^{(\pm)}_e$ is given by

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[5] The invariant vectors under $U^{(+)}$ are non-vanishing for the representations of the type $(0, \rho)$. In this case the invariant vectors can be given explicitly as functions in the representation (A4), namely

$$\mathcal{W}^{(+)0\rho}(z_1, z_2) = (|z_1|^2 + |z_2|^2)^{\frac{i}{2}\rho - 1}. \tag{11}$$

The $U^{(-)}$ case is more subtle, invariant vectors are not zero in the representations of the type $(0, \rho)$, and $(4k, 0)$ $(k = 1, 2, \dots)$(see [26]). The invariant vectors for the representations $(0, \rho)$ are given by

$$\mathcal{W}^{(-)0\rho}(z_1, z_2) = (|z_1|^2 - |z_2|^2)^{\frac{i}{2}\rho - 1} + (|z_2|^2 - |z_1|^2)^{\frac{i}{2}\rho - 1}, \tag{12}$$

while the ones corresponding to representations of the type $(4k, 0)$ $(k = 1, 2, \dots)$ are

$$\mathcal{W}^{(-)4k0}(z_1, z_2) = \delta(|z_1|^2 - |z_2|^2) \left(\frac{z_1}{\bar{z}_2}\right)^{2k}. \tag{13}$$

$$A_e^{(\pm)}(\rho_1,\dots\rho_4;n_1,\dots n_4)=\int d\gamma\ \bar{D}^{n_1\rho_1}_{\mathcal{W}^{(\pm)}\mathcal{W}^{(\pm)}}(\gamma)\dots\bar{D}^{n_4\rho_4}_{\mathcal{W}^{(\pm)}\mathcal{W}^{(\pm)}}(\gamma), \tag{15}$$

and $A_v^{(\pm)}$ by

$$\begin{aligned}&A_v^{(\pm)}(\rho_1,\dots\rho_{10};n_1,\dots n_{10})=\\&\int\prod_{i=1}^{5}d\gamma_i\ \bar{D}^{n_1\rho_1}_{\mathcal{W}^{(\pm)}\mathcal{W}^{(\pm)}}(\gamma_1\gamma_5^{-1})\bar{D}^{n_2\rho_2}_{\mathcal{W}^{(\pm)}\mathcal{W}^{(\pm)}}(\gamma_1\gamma_4^{-1})\bar{D}^{n_3\rho_3}_{\mathcal{W}^{(\pm)}\mathcal{W}^{(\pm)}}(\gamma_1\gamma_3^{-1})\bar{D}^{n_4\rho_4}_{\mathcal{W}^{(\pm)}\mathcal{W}^{(\pm)}}(\gamma_1\gamma_2^{-1})\\&\bar{D}^{n_5\rho_5}_{\mathcal{W}^{(\pm)}\mathcal{W}^{(\pm)}}(\gamma_2\gamma_5^{-1})\bar{D}^{n_6\rho_6}_{\mathcal{W}^{(\pm)}\mathcal{W}^{(\pm)}}(\gamma_2\gamma_4^{-1})\bar{D}^{n_7\rho_7}_{\mathcal{W}^{(\pm)}\mathcal{W}^{(\pm)}}(\gamma_2\gamma_3^{-1})\bar{D}^{n_8\rho_8}_{\mathcal{W}^{(\pm)}\mathcal{W}^{(\pm)}}(\gamma_3\gamma_5^{-1})\bar{D}^{n_9\rho_9}_{\mathcal{W}^{(\pm)}\mathcal{W}^{(\pm)}}(\gamma_3\gamma_4^{-1})\bar{D}^{n_{10}\rho_{10}}_{\mathcal{W}^{(\pm)}\mathcal{W}^{(\pm)}}(\gamma_4\gamma_5^{-1}).\end{aligned} \tag{16}$$

In Fig. (1), each $\bar{D}^{n\rho}_{\mathcal{W}^{(\pm)}\mathcal{W}^{(\pm)}}(\gamma)$ in the previous expressions corresponds to a line bounded by two dark dots. Up to this point we have treated the two theories simultaneously. In the following section we review the properties of the $S^{+}[\phi]$-model introduced in [2], and then we present the new $S^{+}[\phi]$-model.

A. Lorentzian spinfoam over the Lobachevskian space

In this section we summarize the results presented in [2] corresponding to the model defined by the action $S^{+}[\phi]$ above. The functions $\bar{D}^{n\rho}_{\mathcal{W}^{(+)}\mathcal{W}^{(+)}}(\gamma)$ are realized as functions over the hyperboloid $y^\mu y_\mu=1$, $y_0>0$ in Minkowski space, known as Lobachevskian space (here denoted as H^{+}). They are computed in the following section, using the theory of harmonic analysis over H^{+} defined in [26]. However, in this case they can be also computed, in a much more explicit way, using the canonical basis defined in the appendix, and (A27) (this is a consequence of the definition of the canonical basis which is given in terms of functions on $SU(2)$, see [2]). Here we choose the derivation based on [26], because it is easier to extend to the new model in the following subsection. According to equation (40) we have

$$\bar{D}^{n\rho}_{\mathcal{W}^{(+)}\mathcal{W}^{(+)}}(\gamma_1\gamma_2^{-1})=\delta_{n0}K^{+}_{\rho}(\eta(\gamma_1\gamma_2^{-1})):=K^{+}_{\rho}(y_1,y_2), \tag{17}$$

where $K^{+}_{\rho}(\eta)$ is given in (39). Finally, the invariant measure on $SL(2,C)$ is simply the product of the invariant measures of the hyperboloid and $SU(2)$, that is $d\gamma=du^{+}\,dy$. Using all this, the vertex and edge amplitudes can be expressed in simple form. The edge amplitude (15) becomes

$$A_e(\rho_1,\dots\rho_4)=\int dy\ K^{+}_{\rho_1}(y)K^{+}_{\rho_2}(y)K^{+}_{\rho_3}(y)K^{+}_{\rho_4}(y), \tag{18}$$

where we have dropped the n's from our previous notation, since now they all take the value zero. This expression is finite, and its explicit value is computed in [1]. Finally, the vertex amplitude (16) results

$$\begin{aligned}A_v(\rho_1,\dots\rho_{10})=\int dy_1\dots dy_5\ &K^{+}_{\rho_1}(y_1,y_5)K^{+}_{\rho_2}(y_1,y_4)K^{+}_{\rho_3}(y_1,y_3)K^{+}_{\rho_4}(y_1,y_2)\\&K^{+}_{\rho_5}(y_2,y_5)K^{+}_{\rho_6}(y_2,y_4)K^{+}_{\rho_7}(y_2,y_3)K^{+}_{\rho_8}(y_3,y_5)K^{+}_{\rho_9}(y_3,y_4)K^{+}_{\rho_{10}}(y_4,y_5).\end{aligned} \tag{19}$$

The previous amplitude is proportional to the infinite volume of the gauge group $SL(2,C)$. We can now remove this trivial divergence by dropping one of the group integrations (see footnote 3 above). The vertex amplitude (19) is precisely the one defined by Barrett and Crane in [1]. The spinfoam model is finally given by

$$A^{(+)}(J)=\int_{\rho_f}d\rho_f\prod_f\ \rho_f^2\ \prod_e\ A^{+}_e(\rho_{e_1},\dots\rho_{e_4})\ \prod_v\ A^{+}_v(\rho_{v_1},\dots\rho_{v_{10}}), \tag{20}$$

It corresponds to the Lorentzian model presented in [2]. As it was argued there, a more realistic model should include also simple representations of the kind $(n,0)$. In the following subsection we show that this is the case in the theory defined by the action $S^{(-)}[\phi]$.

B. Lorentzian spinfoam over the imaginary Lobachevskian space

Here we introduce the new model defined by the action $S^{(-)}[\phi]$ above. As it is shown in the next section, the matrix elements $\bar{D}^{n\rho}_{\mathcal{W}^{(-)}\mathcal{W}^{(-)}}(\gamma)$ in (15) and (16) are realized as functions on the imaginary Lobachevskian space (in Minkowski

space realized as the 1-sheeted hyperboloid $y^\mu y_\mu = -1$ where the point y is identified with $-y$). They correspond to the irreducible components in the harmonic expansion of the delta distribution on this space. According to equation (63)

$$\bar{D}^{n\rho}_{\mathcal{W}^{(-)}\mathcal{W}^{(-)}}(\gamma_1\gamma_2^{-1}) = K^-_{n\rho}(\eta(\gamma_1\gamma_2^{-1}), \hat{r}_z(\gamma_1\gamma_2^{-1})) := K^-_{n\rho}(y_1, y_2), \tag{21}$$

where $K^-_{n\rho}(\eta, \hat{r}_z)$ is defined as

$$K^-_{n\rho}(\eta, \hat{r}_z) = \delta_{n0} K^-_{\rho}(\eta, \hat{r}_z) + \delta_{4k0}\delta(\rho) K^-_{4k}(\eta, \hat{r}_z), \tag{22}$$

according to (53) and (61). The invariant measure on $SL(2,C)$ is simply the product of the invariant measures of the imaginary Lobachevskian space and $U^{(-)}$ respectively, that is $d\gamma = du^- \, dy$. The edge and vertex amplitudes then become

$$A_e(\rho_1, \ldots \rho_4; n_1, \ldots n_4) = \int dy \; K^-_{n_1\rho_1}(y) K^-_{n_2\rho_2}(y) K^-_{n_3\rho_3}(y) K^-_{n_4\rho_4}(y), \tag{23}$$

and

$$\begin{aligned} A^-_v(\rho_1, \ldots \rho_{10}; n_1, \ldots n_{10}) = \int dy_2 \ldots dy_5 \; & K^-_{n_1\rho_1}(y_1, y_5) K^-_{n_2\rho_2}(y_1, y_4) K^-_{n_3\rho_3}(y_1, y_3) K^-_{n^4\rho_4}(y_1, y_2) \\ & K^-_{n_5\rho_5}(y_2, y_5) K^-_{n_6\rho_6}(y_2, y_4) K^-_{n_7\rho_7}(y_2, y_3) K^-_{n_8\rho_8}(y_3, y_5) K^-_{n_9\rho_9}(y_3, y_4) K^-_{n_10\rho_{10}}(y_4, y_5) \end{aligned} \tag{24}$$

respectively. Where as in the previous model we have dropped one of the integration variables from the definition of the vertex amplitude to remove a trivial infinite volume factor. In the spinfoam model defined by $S^{(-)}[\phi]$ the amplitude of a given 2-complex J is finally

$$A^{(-)}(J) = \sum_{n_f} \int_{\rho_f} d\rho_f \prod_f \, (\rho_f^2 + n_f^2) \; \prod_e \; A^-_e(\rho_{e_1}, \ldots \rho_{e_4}; n_{e_1}, \ldots n_{e_4}) \; \prod_v \; A^-_v(\rho_{v_1}, \ldots \rho_{v_{10}}; n_{v_1}, \ldots n_{v_{10}}). \tag{25}$$

III. CALCULATION OF THE KERNELS $K^\pm$

In this section we compute the values of the projector $K^\pm$ which appear in the vertex and edge amplitudes of the two models.

A. Functions on the Lobachevskian space

Take the 2-sheeted hyperboloid on Minkowski space defined by $x^\nu x_\nu = 1$. Every point on upper sheet of the hyperboloid can be written as

$$x_g = g g^\dagger. \tag{26}$$

This space is a possible realization of the Lobachevskian space (from now on denoted by H^+). The theory of harmonic analysis on this space is relevant for the spinfoam model defined by the action $S^{(+)}[\phi]$ as can be seen from the fact that $H^+ = SL(2,C)/SU(2)$. Given a square integrable function $\tilde{f}(g)$ over $SL(2,C)$ we can define a square integrable function $f(x) \in \mathcal{L}^2(H^+)$ by means of averaging $\tilde{f}$ over the subgroup $U^{(+)} = SU(2)$. Namely,

$$f(x) = \int_{U^{(+)}} \tilde{f}(g_x u^{(+)}) du^{(+)}, \tag{27}$$

where $g_x \in SL(2,C)$ represents the equivalence class of transformations taking the apex $(1,0,0,0)$ to the point x on the hyperboloid. Expanding $\tilde{f}(g)$ in modes, using (A24), we can construct the theory of harmonic analysis of $f(x)$. Thus

$$f(x) = \sum_{n=0}^{\infty} \int_{\rho=0}^{\infty} \int_{U^{(+)}} \bar{D}^{n,\rho}_{j_1 q_1 j_2 q_2}(g_x u^{(+)}) \tilde{f}^{j_1 q_1 j_2 q_2}_{n,\rho} (n^2 + \rho^2) d\rho du^{(+)}, \tag{28}$$

then using (10) and defining $f^{j_1q_1}_{n,\rho} := \tilde{f}^{j_1q_1jq}_{n,\rho}\bar{\mathcal{W}}^{(+)n,\rho}_{jq}$ we obtain

$$f(x) = \sum_{n=0}^{\infty}\int_{\rho=0}^{\infty}\left[\bar{D}^{n,\rho}_{j_1q_1j_2q_2}(g_x)\mathcal{W}^{(+)n,\rho}_{j_2q_2}\right] f^{j_1q_1}_{n,\rho}(n^2+\rho^2)d\rho. \tag{29}$$

Finally using (A25),(10), and the fact that any $g \in SL(2,C)$ can be written as $g_x u^{(+)}$

$$\begin{aligned} f^{j_1q_1}_{n,\rho} &= \int \tilde{f}(g_x u^{(+)}) D^{n,\rho}_{j_1q_1jq}(g_x u^{(+)})\bar{\mathcal{W}}^{(+)n,\rho}_{j_2q_2} d(g_x u^{(+)}) \\ &= \int \tilde{f}(g_x u^{(+)}) D^{n,\rho}_{j_1q_1jq}(g_x)\bar{\mathcal{W}}^{(+)n,\rho}_{j_2q_2} dx du^{(+)} \\ &= \int f(x)\left[D^{n,\rho}_{j_1q_1jq}(g_x)\bar{\mathcal{W}}^{(+)n,\rho}_{jq}\right] dx. \end{aligned} \tag{30}$$

In the second line we used $d(g_x u^{(+)}) = dx du^{(+)}$, where dx is the invariant measure on $H^{(+)}$ and invariance of $\mathcal{W}^{(+)n,\rho}$ under U^+. In the last line we used equation (27). Combining these two equations we can write

$$\delta(x) = \sum_{n=0}^{\infty}\int_0^{\infty}(n^2+\rho^2)d\rho\ \ \bar{\mathcal{W}}^{(+)n,\rho}_{j_1q_1}\bar{D}^{n,\rho}(g_x)_{j_1q_1j_2q_2}\mathcal{W}^{(+)n,\rho}_{j_2q_2}. \tag{31}$$

The term in brackets in (29) and (30) corresponds to the analogous of spherical harmonics in the case of functions over $S^2 = SU(2)/U(1)$. They are different from zero only for simple irreducible representations of the type $(0,\rho)$. This can be explicitly shown using the properties of the canonical basis defined in the appendix (see also [2]). However, this method can not be easily extended to the case which is relevant to the new model introduced in the paper. We therefore complete the construction in the more general framework of reference [26], where the theory of harmonic analysis over the $SL(2,C)$ homogeneous spaces is defined with the aid of integral geometry elements.

Given $f(x) \in \mathcal{L}^2(H^+)$, square integrable function over the Lobachevskian space, its Gelfand transform is defined as

$$F(\xi;\rho) = \int_{H^-} f(x)\,(x^\nu\xi_\nu)^{i\rho/2-1}\,dx, \tag{32}$$

where $\rho \in [0,\infty)$ and ξ is a vector on the null cone C^+ normalized such that $\xi_0 = 1$. The invariant measure dx on the hyperboloid is defined up to a constant factor that we choose in order to simplify some equations (our measure differs from the one in [26] by a $(4\pi)^2$ factor). It turn out that the function $F(\xi;\rho)$ lives in the irreducible representation of the type $(0,\rho)$ of $SL(2,C)$. Moreover, it corresponds to the Fourier component of $f(x)$ in the balance representation $(0,\rho)$ as can be explicitly seen writing the inversion formula for (32). Choosing coordinates this inversion formula can be written as

$$f(x) = \int_0^{\infty}\rho^2 d\rho \int_{C^+} F(\xi;\rho)\,(x^\nu\xi_\nu)^{-i\rho/2-1}\,d\omega, \tag{33}$$

where

$$d\omega = \frac{1}{4\pi}\sin(\theta)d\theta d\phi, \tag{34}$$

is the normalized measure on the sphere defined by the null cone with normalization $\xi_0 = 1$. Combining (33) with (32) the resolution of the identity on H^+ becomes

$$\delta(x,y) = \int_0^{\infty}\rho^2 d\rho \int_{\Gamma^+}(y^\nu\xi_\nu)^{i\rho/2-1}\,(x^\nu\xi_\nu)^{-i\rho/2-1}\,d\omega \tag{35}$$

The previous equation can be express as a sum over the projectors $K^{(+)}_\rho(x,y)$ over the representation $(0,\rho)$, namely

$$\delta(x,y) = \int_0^{\infty}\rho^2 d\rho K_\rho(x,y), \tag{36}$$

where

$$K^{(+)}_\rho(x,y) = \int_{\Gamma^+} (y^\nu \xi_\nu)^{i\rho/2-1} (x^\nu \xi_\nu)^{-i\rho/2-1}\, d\omega. \tag{37}$$

It is easy to see that the previous function depends only on the hyperbolic distance between x and y. To compute explicitly its value we chose x to be the hyperboloid apex ($x = (1,0,0,0)$), while we take $y = (\cosh(\eta), 0, 0, \sinh(\eta))$. In our normalization the 4-vector ξ can be written as

$$\xi = (1, \sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))\,. \tag{38}$$

With all this the previous equation becomes

$$\begin{aligned} K^{(+)}_\rho(\eta) &= \frac{1}{4\pi}\int (\cosh(\eta) - \cos(\theta)\sinh(\eta))^{i\rho/2-1} \sin(\theta) d\theta d\phi \\ &= \frac{2\,\sin(\frac{1}{2}\rho\eta)}{\rho\,\sinh(\eta)} \end{aligned} \tag{39}$$

Comparing equation (31) with (36) we conclude that

$$D^{n,\rho}_{\mathcal{W}^{(+)}\mathcal{W}^{(+)}}(g_y) := \bar{\mathcal{W}}^{(+)n,\rho}_{j_1q_1} \bar{D}^{n,\rho}(g_y)_{j_1q_1j_2q_2} \mathcal{W}^{(+)n,\rho}_{j_2q_2} = \delta_{n0} K^{(+)}_\rho(x,y) \tag{40}$$

B. Functions on the imaginary Lobachevskian space

The 1-sheeted hyperboloid on Minkowski space is given by the points such that $x^\nu x_\nu = -1$. Every point on the hyperboloid can be written as

$$x_g = g\sigma_3 g^\dagger. \tag{41}$$

The imaginary Lobachevskian space H^- corresponds to the 1-sheeted hyperboloid where the point x is identified with $-x$. This is exactly the homogeneous space $SL(2,C)/U^{(-)}$ relevant for the construction of the new model presented in the paper. In analogy to the previous section we can defined a function on the imaginary Lobachevskian space by means of averaging functions on $SL(2,C)$ over the sub-group $U^{(-)} = SU(1,1) \times Z_2$. Given $\tilde{f}(g) \in \mathcal{L}^2(SL(2,C))$ we can define $f(x) \in \mathcal{L}^2(H^{(-)})$ as

$$f(x) = \int_{U^{(-)}} \tilde{f}(g_x u^{(-)}) du^{(-)}, \tag{42}$$

where $g_x \in SL(2,C)$ represents the equivalence class of transformations taking the point $(0,0,0,1)$ to the point x on the hyperboloid. The difficulty now is that the sub-group $U^{(-)}$ is no longer compact. As a consequence the RHS of the previous equation is not a square integrable as a function on $SL(2,C)$. Expanding $\tilde{f}(g)$ in modes, using (A24),

$$f(x) = \sum_{n=0}^{\infty} \int_{\rho=0}^{\infty} \int_{U^{(-)}} \bar{D}^{n,\rho}_{j_1q_1j_2q_2}(g_x u^{(-)}) \tilde{f}^{j_1q_1j_2q_2}_{n,\rho} (n^2+\rho^2) d\rho du^{(-)}, \tag{43}$$

then using (10) and defining $f^{j_1q_1}_{n,\rho} := \tilde{f}^{j_1q_1jq}_{n,\rho} \mathcal{W}^{(-)n,\rho}_{jq}$ we obtain

$$f(x) = \sum_{n=0}^{\infty} \int_{\rho=0}^{\infty} \left[\bar{D}^{n,\rho}_{j_1q_1j_2q_2}(g_x) \mathcal{W}^{(-)n,\rho}_{j_2q_2}\right] f^{j_1q_1}_{n,\rho} (n^2+\rho^2) d\rho, \tag{44}$$

where

$$\begin{aligned} f^{j_1q_1}_{n,\rho} &= \int \tilde{f}(g_x u^{(-)}) D^{n,\rho}_{j_1q_1jq}(g_x u^{(-)}) \bar{\mathcal{W}}^{(-)n,\rho}_{j_2q_2} d(g_x u^{(-)}) \\ &= \int \tilde{f}(g_x u^{(-)}) D^{n,\rho}_{j_1q_1jq}(g_x) \bar{\mathcal{W}}^{(-)n,\rho}_{j_2q_2} dx du^{(-)} \\ &= \int f(x) \left[D^{n,\rho}_{j_1q_1jq}(g_x) \bar{\mathcal{W}}^{(-)n,\rho}_{jq}\right] dx. \end{aligned} \tag{45}$$

The non-compactness of $U^{(-)}$ implies the non-normalizability of the $\mathcal{W}^{(-)n,\rho}$ invariant vectors. Their presence in (A24) correspond to a distributional factor in the Fourier components of $f(x)$ thought as a $SL(2,C)$ function. [6] Combining the last two equations we can read off the expression for the delta distribution on $H^{(-)}$, namely

$$\delta(x) = \sum_{n=0}^{\infty} \int_0^{\infty} (n^2+\rho^2) d\rho \;\; \bar{\mathcal{W}}^{(-)n,\rho}_{j_1 q_1} \bar{D}^{n,\rho}(g_x)_{j_1 q_1 j_2 q_2} \mathcal{W}^{(-)n,\rho}_{j_2 q_2}. \tag{46}$$

Now we present the results of [26] on the harmonic analysis on H^- based on Gelfand's integral transforms. Given $f(x) \in \mathcal{L}^2(H^-)$, square integrable function over the imaginary Lobachevskian space, it can be expanded in terms of its irreducible $SL(2,C)$ components as

$$\begin{aligned} f(x) = \int_0^{\infty} \rho^2 d\rho \int_{\Gamma^+} F(\xi;\rho) \left| x^\nu \xi_\nu \right|^{-i\rho/2-1} d\omega \; + \\ 32\,\pi \sum_{k=1}^{\infty} (4k)^2 \int_{C^+} F(\xi,x;2k) \delta(x^\nu \xi_\nu) d\omega, \end{aligned} \tag{47}$$

where C^+, and $d\omega$ are defined as in the previous section. The functions $F(\xi;\rho)$ and $F(\xi,x;2k)$ correspond to the Fourier components in the balance representation $(0,\rho)$ and $(4k,0)$ respectively. They are explicitly given by

$$F(\xi;\rho) = \int_{H^-} f(x) \left| x^\nu \xi_\nu \right|^{i\rho/2-1} dx, \tag{48}$$

and

$$F(\xi,x;2k) = \frac{1}{k} \int_{H^-} f(y) e^{-2ik\Theta(x,y)} \delta(y^\nu \xi_\nu) dy, \tag{49}$$

where the function $\Theta(x,y)$ is defined by the following equation

$$\cos(\Theta) = \left| x^\nu y_\nu \right|. \tag{50}$$

The geometric interpretation of the angle Θ defined in the previous equation can be done as follows. The line generators of the 1-sheeted hyperboloid correspond to null geodesics known as isotropic lines [26]. For each point

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[6] A simple analogy corresponds to the following example in Fourier analysis on $\Re^2$. Take the square integrable function $\tilde{f}(x,y) = \exp(-x^2-y^2)$. The analogous to the group $SL(2,C)$ is here the group of translations in $\Re^2$. Every function $\tilde{f}(x,y)$ can be thought of as a function of the group element that takes the origin into the point (x,y). We can define an invariant function under the action of the subgroup U of translations in the y direction by averaging $\tilde{f}(x,y)$ under the action of U, namely

$$f(x) = \int_{-\infty}^{\infty} \tilde{f}(x, y+u) du = \sqrt{\pi} e^{-x^2},$$

where in the last equality we have used our Gaussian example for $\tilde{f}(x,y)$. The function $f(x)$ is a perfectly square integrable function of one variable but is no longer square integrable on $\Re^2$. Writing the previous equation in momentum space we have

$$\begin{aligned} f(x) &= \frac{1}{4\pi^2} \int \tilde{F}(k_x,k_y) e^{i(xk_x+(y+u)k_y)} du dk_x dk_y \\ &= \frac{1}{2\pi} \int \tilde{F}(k_x,0) e^{ixk_x} dk_x, \end{aligned}$$

where

$$\tilde{F}(k_x,0) = \int \tilde{f}(x,y) e^{-ixk_x} dx dy = \int f(x) e^{-ixk_x} dx$$

is the Fourier component of $f(x)$ in one dimension. If we think of $f(x)$ as a function on $\Re^2$ then its Fourier transform takes distributional values in k_y, namely $F(k_x,0)\delta(k_y)$.

$x \in H^-$ the null geodesic $x(\lambda) = x + \lambda\xi$ for $\lambda \in \Re$ and $\xi \in C^+$ such that $x^\mu\xi_\mu = 0$ is on H^{-1} for all values of λ. Fixing x there is a circle worth of such lines. Consider a second point y and search for an isotropic line containing y and parallel to one crossing x (i.e., having the same null generator ξ). In order for it for exist, we need $x^\mu\xi_\mu = y^\mu\xi_\mu = 0$. In general, the two lines will intersect the sphere given by the section $x_0 = 0$ of the hyperboloid in two different points. One can easily verify that the scalar product $x(\lambda_1)^\mu y_\mu(\lambda_2)$ is independent of the values of λ_1 and λ_2. Therefore we can calculate the scalar product of eq.(50) at the λ-values for which the two lines intersect the sphere. By doing that we conclude that the value of $\Theta(x,y)$ corresponds to the azimuthal separation of those points on the sphere. Combining (47) with previous equations the resolution of the identity on H^- becomes

$$\delta(x,y) = \int_0^\infty \rho^2 d\rho \int_{\Gamma^+} |y^\nu\xi_\nu|^{i\rho/2-1} |x^\nu\xi_\nu|^{-i\rho/2-1} d\omega + \\ 32\,\pi \sum_{k=1}^\infty (4k)^2 \int_{\Gamma^+} e^{-2ik[\Theta(x,y)]}\delta(x^\nu\xi_\nu)\delta(y^\nu\xi_\nu) d\omega. \tag{51}$$

The previous equation can be express as a "sum" over the projectors $K_\rho^{(-)}(x,y)$ and $K_{4k}^{(-)}(x,y)$, namely

$$\delta(x,y) = \int_0^\infty \rho^2 d\rho K_\rho^{(-)}(x,y) + \sum_{k=1}^\infty (4k)^2 K_{4k}^{(-)}(x,y), \tag{52}$$

where

$$K_\rho^{(-)}(x,y) = \int_{\Gamma^+} |y^\nu\xi_\nu|^{i\rho/2-1} |x^\nu\xi_\nu|^{-i\rho/2-1} d\omega, \tag{53}$$

and

$$K_{4k}^{(-)}(x,y) = \frac{32\,\pi}{k} \int_{\Gamma^+} e^{-2ik[\Theta(x,y)]}\; \delta(x^\nu\xi_\nu)\; \delta(y^\nu\xi_\nu) d\omega. \tag{54}$$

Lets analyze equation (53); we take $\xi = (1, \sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))$, and $x = (0,0,0,1)$. We parameterize y in the following way

$$y = (\sinh(\eta), \cosh(\eta)\; \hat{r})\,, \tag{55}$$

where $\hat{r}$ represents a point on the unit sphere. In terms of these coordinates (53) becomes

$$\begin{aligned} &K_\rho^{(-)}(\eta,\hat{r}) = \frac{1}{4\pi}\int \sin(\theta)\, d\theta d\phi \,\times \\ &|\sinh(\eta) - \cosh(\eta)\,[\sin(\theta)\cos(\phi)\hat{r}_x + \sin(\theta)\sin(\phi)\hat{r}_y + \cos(\theta)\hat{r}_z]|^{i\rho/2-1}\, |\cos(\theta)|^{-i\rho/2-1} = \\ &\int dt d\phi \left|\sinh(\eta) - \cosh(\eta)\left[(1-t^2)^{(1/2)}\left(\cos(\phi)\hat{r}_x + \sin(\phi)\hat{r}_y\right) + t\hat{r}_z\right]\right|^{i\rho/2-1} |t|^{-i\rho/2-1}\,. \end{aligned} \tag{56}$$

Finally using that $A\,\sin(\alpha) + B\,\cos(\alpha) = (A^2+B^2)^{1/2}\sin(\alpha+\alpha_0)$ we obtain

$$K_\rho^{(-)}(\eta,\hat{r}) = \frac{1}{4\pi}\int dt d\phi \left|\sinh(\eta) - \cosh(\eta)\left[(1-t^2)^{(1/2)}(1-\hat{r}_z^2)^{(1/2)}\sin(\phi) + t\hat{r}_z\right]\right|^{i\rho/2-1} |t|^{-i\rho/2-1}\,. \tag{57}$$

From the previous equation we conclude that $K_\rho^{(-)}(\eta,\hat{r})$ behaves asymptotically as

$$K_\rho^{(-)}(\eta,\hat{r}) = \tilde{\alpha}_\rho(\hat{r}_z)\; e^{-\eta}\, e^{i\eta\rho/2}, \tag{58}$$

where $\tilde{\alpha}(\hat{r}_z)$ is an integral of a finite function over the compact space $[0,1]\times S^1$ and therefore is finite. When $\hat{r}_z = 1$ the integral above can be performed and we obtain

$$K_\rho^{(-)}(\eta, \hat{r}_z = 1) = \frac{2\sin(\frac{1}{2}\rho\eta)}{\rho\,\sinh(\eta)}. \tag{59}$$

When the points x and y lay on the same boost orbit the projector $K_\rho^{(-)}(x,y)$ has the same form as $K_\rho^{(+)}(x,y)$. In the same parametrization the projector $K_{4k}^{(-)}(x,y)$ becomes

$$K^{(-)}_{4k}(\eta,\hat{r}_z) = \frac{8\,e^{-2ik\Theta(\eta,\hat{r}_z)}}{k}\int \delta\left(\sinh(\eta)-\cosh(\eta)(1-\hat{r}_z^2)^{(1/2)}\sin(\phi)\right)d\phi$$
$$= \frac{8\,e^{-2ik\Theta(\eta,\hat{r}_z)}}{k\left(1-\hat{r}_z^2\cosh^2(\eta)\right)^{1/2}}\int \delta\left(\phi-\arcsin\left[\frac{\tanh(\eta)}{(1-\hat{r}_z^2)^{(1/2)}}\right]\right)d\phi. \tag{60}$$

The integral on the right vanishes unless $\left|\frac{\tanh(\eta)}{(1-\hat{r}_z^2)^{(1/2)}}\right| \leq 1$. Notice that therefore $K^{(-)}_{4k}(\eta,\hat{r}_z)$ has support on the values of η, and $\hat{r}_z$ for which $\Theta = \arccos(|\cosh(\eta)\hat{r}_z|)$ is real with a range $0 \leq \Theta \leq \frac{\pi}{2}$. Using that $\cos(\Theta) = |\cosh(\eta)\hat{r}_z|$ (equation (50)) we finally obtain

$$K^{(-)}_{4k}(\eta,\hat{r}_z) = \frac{8\,e^{-2ik\Theta}}{k\,\sin(\Theta)} \tag{61}$$

for $0 \leq \Theta(\eta,\hat{r}_z) \leq \frac{\pi}{2}$ and zero otherwise. The real part of the previous projector diverges at $\Theta = 0$, i.e., on the curve $\hat{r}_z = \pm\cosh^{-1}(\eta)$. If we define $J = k - \frac{1}{2}$ then J takes all the half-integer values and the imaginary part of $K^{(-)}_{4k}(\eta,\hat{r}_z)$ becomes

$$\mathrm{Im}\left[K^{(-)}_{4k}(\eta,\hat{r}_z)\right] = \frac{8}{J+\frac{1}{2}}\;\frac{\sin\left((2J+1)\Theta\right)}{\sin(\Theta)} \tag{62}$$

for $0 \leq \Theta \leq \pi$ and vanishes otherwise. The imaginary part of the projector on the discrete representations has the same form as the one appearing in the Euclidean Barrett-Crane models. Finally, comparing (46) with (52) for $x = (0,0,0,1)$ we conclude that

$$D^{n,\rho}_{\mathcal{W}^{(-)}\mathcal{W}^{(-)}}(g_y) = \delta_{n0}K^{(-)}_{\rho}(x,y) + \delta_{n,4k}\delta(\rho)K^{(-)}_{4k}(x,y). \tag{63}$$

IV. DISCUSSION

We have carried over a generalization of the model defined in [2]. The new model is given by an $SL(2,C)$ BF quantum theory plus a quantum implementation of the constraints that reduce BF theory to Lorentzian general relativity. This corresponds to the restriction to simple representations, those for which the Casimir $\hat{\mathcal{C}}_2$ vanishes (see (A31)). In contrast with the previous model, the present one includes also elements of the discrete series in the set of simple representations.

Four dimensional Lorentzian quantum spacetime appears as a fully combinatorial notion represented by spinfoams colored by simple representations of $SL(2,C)$. The model possesses an intrinsically defined local causal structure with is non-perturbative and background independent. Causality in the model is induced by the algebra of $SL(2,C)$ (see equations (A30) and (A31)). In particular, the Casimir (A30) can be interpreted as the square of the area operator corresponding to quantized bivectors [5]. Space-like and time-like bivectors can be classified according to the two possibilities $\hat{\mathcal{C}}_1 > 0$ or $\hat{\mathcal{C}}_1 < 0$. We define a space-like section of a given spinfoam as the colored graph (spin network) defined by the intersection of a 3-surface with the corresponding 2-complex such that it is labeled by simple representations in the discrete series. On this representations the area operator $\hat{\mathcal{A}}$ reduces to

$$\hat{\mathcal{A}} \;\sim\; \sqrt{J(J+1)}\;\hat{1}, \tag{64}$$

where $J := k - 1/2$ ($k = 1, 2 \ldots$) and therefore takes only half-integer values. Notice that the spectrum of the area on spatial sections of the spinfoams of the model is contained in the one predicted by Loop Quantum Gravity in the canonical formalism [22]. [7] Only those eigenvalues corresponding to half integer spin appear. This is associated to the fact that the model was based on the harmonic analysis of even functions on the one sheeted hyperboloid, the

[7] Our definition of space-like section should be in agreement with the geometrical analysis of the discretization of BF theory with the corresponding passage to GR through the implementation of Plebanski's constraints. As it is pointed out in [1] this is a delicate issue. To answer this question a rigorous definition of the area operator as well as a deeper understanding of the geometry of the quantization prescription is needed. This important issue will be study in the future.

imaginary Lobachevkian space. The extension of the model including the other part of the spectrum seems possible and it will be study elsewhere.

A necessary condition for the model to be well defined is the finiteness of the edge and vertex amplitudes appearing in (25). Both amplitudes turn out to be finite in the $S^+[\phi]$ model [24]. We do not address this issue in the paper, but we want to make a few comments on some of the results that might be relevant for future research in this respect. In the new model the projectors $K_\rho^{(-)}(x,y)$ have the same asymptotic behavior of $K_\rho^{(+)}(x,y)$ (they even coincide when the two points are located in the same spatial direction (see eq. (59)). Therefore, one would expect no divergences coming from this kind of projectors. The analysis of the discrete projectors $K_{4k}^{(-)}(x,y)$ is more delicate. The imaginary part

$$\mathrm{Im}\left[K_{4k}^{(-)}(x,y)\right] \sim \frac{\sin\left((2J+1)\Theta(x,y)\right)}{\sin(\Theta(x,y))} \tag{65}$$

is well behaved and has the same functional form of the corresponding projector in the Euclidean Barrett-Crane model [25]. The real part diverges on $\Theta = 0$ which according to the geometrical interpretation given in the last section corresponds to the situation in which x and y lay on the same null generator of the hyperboloid. The study of the convergence of the different amplitudes appearing in the new model is left for future studies.

Another problem is the the convergence of the sum over representations in (25). This problem appears also in the Euclidean models. To cure it, the Barrett-Crane model was defined in terms of a quantum deformation of the gauge group ($SO_q(4)$, with $q^n = 1$). The quantum deformation introduces a cut-off in the summ over representations that regularizes the amplitudes. In the limit in which the quantum deformation is removed ($q \to 1$), divergences appear whenever the 2-complex J includes bubbles. A similar regularization for the Lorentzian state sum model is suggested by Barrett and Crane in [1]. A different strategy for dealing with this infinity was suggested in reference [14], using the field theory over group technology. In this reference, we have defined a natural variant of the Euclidean Barrett-Crane model based on a different implementation of the BF-to-GR constraints which, however, turns out to be finite [23]. On this model, see also [30]. The Lorentzian model presented in [2] as well as its extension presented here correspond to the finite version of the Euclidean model. Accordingly, although further study is certainly needed, we suspect that the Lorentzian models presented here might also be finite.

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APPENDIX A: REPRESENTATION THEORY OF $SL(2,C)$

We review a series of relevant facts about $SL(2,C)$ representation theory. Most of the material of this section can be found in [26,27]. For a very nice presentation of the subject see also [28].

We denote an element of $SL(2,C)$ by

$$g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, \tag{A1}$$

with α, β, γ, δ complex numbers such that $\alpha\delta - \beta\gamma = 1$. All the finite dimensional irreducible representations of $SL(2,C)$ can be cast as a representation over the set of polynomials of two complex variables z_1 and z_2, of order $n_1 - 1$ in z_1 and z_2 and of order $n_2 - 1$ in $\bar{z}_1$ and $\bar{z}_2$. The representation is given by the following action

$$T(g)P(z_1, z_2) = P(\alpha z_1 + \gamma z_2, \beta z_1 + \delta z_2). \tag{A2}$$

The usual spinor representations can be directly related to these ones.

The infinite dimensional representations are realized over the space of homogeneous functions of two complex variables z_1 and z_2 in the following way. A function $f(z_1, z_2)$ is called homogeneous of degree (a, b), where a and b are complex numbers differing by an integer, if for every $\lambda \in C$ we have

$$f(\lambda z_1, \lambda z_2) = \lambda^a \bar{\lambda}^b f(z_1, z_2), \tag{A3}$$

where a and b are required to differ by an integer in order to $\lambda^a \bar{\lambda}^b$ be a singled valued function of λ. The infinite dimensional representations of $SL(2,C)$ are given by the infinitely differentiable functions $f(z_1, z_2)$ (in z_1 and z_2 and

their complex conjugates) homogeneous of degree $(\frac{\mu+n}{2}-1, \frac{\mu-n}{2}-1)$, with n an integer and μ a complex number. The representations are given by the following action

$$T_{n\mu}(g)f(z_1,z_2) = f(\alpha z_1 + \gamma z_2, \beta z_1 + \delta z_2). \tag{A4}$$

One simple realization of these functions is given by the functions of one complex variables defined as

$$\phi(z) = f(z,1). \tag{A5}$$

On this set of functions the representation operators act in the following way

$$T_{n\mu}(g)\phi(z) = (\beta z+\delta)^{\frac{\mu+n}{2}-1}(\bar\beta\bar z+\bar\delta)^{\frac{\mu-n}{2}-1}\phi\left(\frac{\alpha z+\gamma}{\beta z+\delta}\right). \tag{A6}$$

Two representations $T_{n_1\mu_2}(g)$ and $T_{n_1\mu_2}(g)$ are equivalent if $n_1=-n_2$ and $\mu_1=-\mu_2$.

Unitary representations of $SL(2,C)$ are infinite dimensional. They are a subset of the previous ones corresponding to the two possible cases: μ purely imaginary ($T_{n,i\rho}(g)$ $\mu=i\rho$, $\rho=\bar\rho$, known as the *principal series*), and $n=0$, $\mu=\bar\mu=\rho$, $\rho\neq 0$ and $-1<\rho<1$ ($T_{0\rho}(g)$the *supplementary series*). From now on we concentrate on the principal series unitary representations $T_{ni\rho}(g)$ which we denote simply as $T_{n\rho}(g)$ (dropping the i in front of ρ). The invariant scalar product for the principal series is given by

$$(\phi,\psi) = \int \bar\phi(z)\psi(z)dz, \tag{A7}$$

where dz denotes $dRe(z)dIm(z)$.

There is a well defined measure on $SL(2,C)$ which is right-left invariant and invariant under inversion (namely, $dg = d(gg_0) = d(g_0g) = d(g^{-1})$). Explicitly, in terms of the components in (A1)

$$dg = \left(\frac{i}{2}\right)^3 \frac{d\beta d\gamma d\delta}{|\delta|^2} = \left(\frac{i}{2}\right)^3 \frac{d\alpha d\gamma d\delta}{|\gamma|^2} = \left(\frac{i}{2}\right)^3 \frac{d\beta d\alpha d\delta}{|\beta|^2} = \left(\frac{i}{2}\right)^3 \frac{d\beta d\gamma d\alpha}{|\alpha|^2}, \tag{A8}$$

where $d\alpha$, $d\beta$, $d\gamma$, and $d\delta$ denote integration over the real and imaginary part respectively.

Every square-integrable function, i.e, $f(g)$ such that

$$\int |f(g)|^2 dg \leq \infty, \tag{A9}$$

has a well defined Fourier transform defined as

$$F(n,\rho) = \int f(g)T_{n,\rho}(g)dg. \tag{A10}$$

This equation can be inverted to express $f(g)$ in terms of $T_{n,\rho}(g)$. This is known as the Plancherel theorem which generalizes the Peter-Weyl theorem for finite dimensional unitary irreducible representations of compact groups as $SU(2)$. Namely, every square-integrable function $f(g)$ can be written as

$$f(g) = \frac{1}{8\pi^4}\sum_n \int \mathrm{Tr}[F(n,\rho)T_{n,\rho}(g^{-1})](n^2+\rho^2)d\rho, \tag{A11}$$

where only components corresponding to the principal series are summed over (not all unitary representations are needed) [8] , and

$$\mathrm{Tr}[F(n,\rho)T_{n,\rho}(g^{-1})] = \int \mathcal{F}_{n\rho}(z_1,z_2)\mathcal{T}_{n\rho}(z_2,z_1;g)dz_1dz_2. \tag{A12}$$

[8] If the function $f(g)$ is infinitely differentiable of compact support then it can be shown that $F(n,\rho)$ is an analytic function of ρ and an expansion similar to (A11) can be written in terms of non-unitary representations.

$\mathcal{F}_{n\rho}(z_1, z_2)$, and $\mathcal{T}_{n\rho}(z_2, z_1; g)$ correspond to the kernels of the Fourier transform and representation respectively defined by their action on the space of functions $\phi(z)$ (they are analogous to the momenta components and representation matrix elements in the case of finite dimensional representations), namely

$$F(n,\rho)\phi(z) := \int f(g) T_{n\rho}(g)\phi(z) dg := \int \mathcal{F}_{n\rho}(z,\tilde{z})\phi(\tilde{z}) d\tilde{z}, \tag{A13}$$

and

$$T_{n,\rho}(g)\phi(z) := \int \mathcal{T}_{n\rho}(z,\tilde{z};g)\phi(\tilde{z}) d\tilde{z}. \tag{A14}$$

From (A6) we obtain that

$$\mathcal{T}_{n\rho}(z,\tilde{z};g) = (\beta z+\delta)^{\frac{\rho+n}{2}} (\bar{\beta}\bar{z}+\bar{\delta})^{\frac{\rho-n}{2}} \delta\left(\tilde{z} - \frac{\alpha z+\gamma}{\beta z+\delta}\right). \tag{A15}$$

The resolution of the identity takes the form

$$\delta(g) = \frac{1}{8\pi^4} \sum_n \int \mathrm{Tr}[T_{n,\rho}(g)](n^2+\rho^2)\ d\rho. \tag{A16}$$

a. The canonical basis

There exists an alternative realization of the representations in terms of the space of homogeneous functions $f(z_1, z_2)$ defined above [27]. Because of homogeneity (A3) any $f(z_1, z_2)$ is completely determined by its values on the sphere S^3

$$|z_1|^2 + |z_2|^2 = 1. \tag{A17}$$

As it is well now there is an isomorphism between S^3 and $SU(2)$ given by

$$u = \begin{bmatrix} \bar{z}_2 & -\bar{z}_1 \\ z_1 & z_2 \end{bmatrix} \tag{A18}$$

for $u \in SU(2)$ and z_i satisfying (A17). Alternatively we can define the function $\phi(u)$ of $u \in SU(2)$ as

$$\phi(u) := f(u_{21}, u_{22}), \tag{A19}$$

with f as in (A3). Due to (A3) $\phi(u)$ has the following "gauge" behavior

$$\phi(\gamma u) = e^{i\omega(a-b)}\phi(u) = e^{i\omega n}\phi(u), \tag{A20}$$

for $\gamma = \begin{bmatrix} e^{i\omega} & 0 \\ 0 & e^{-i\omega} \end{bmatrix}$. The action of $T_{n\rho}(g)$ on $\phi(u)$ is induced by its action on $f(z_1, z_2)$ (A4). We can now use Peter-Weyl theorem to express $\phi(u)$ in terms of irreducible representations $D^j_{q_1 q_2}(u)$ of $SU(2)$. However, due to (A20) only the functions $\phi^j_q(u) = (2j+1)^{1/2} D^j_{nq_2}(u)$ are needed (where $j = |n| + k$, $k = 0, 1, \ldots$). Therefore $\phi(u)$ can be written as

$$\phi(u) = \sum_{j=n}^{\infty} \sum_{q=-j}^{j} d^j_q \ \ \phi^j_q(u). \tag{A21}$$

This set of functions is known as the canonical basis. This basis is better suited for generalizing the Euclidean spin foam models, since the notation maintains a certain degree of similarity with the one in [16,14]. We can use this basis to write the matrix elements of the operators $T_{n,\rho}(g)$, namely

$$D^{n\rho}_{j_1 q_1 j_2 q_2}(g) = \int_{SU(2)} \bar{\phi}^{j_1}_{q_1}(u) \ \left[T_{n\rho}(g)\phi^{j_2}_{q_2}(u)\right] du. \tag{A22}$$

Since $T_{n_1 n_2}(u_0)\phi(u) = \phi(uu_0)$, invariance of the $SU(2)$ Haar measure implies that

$$D^{n\rho}_{j_1 q_1 j_2 q_2}(u_0) = \delta_{j_1 j_2}\ D^{j_1}_{q_1 q_2}(u_0). \tag{A23}$$

In terms of these matrix elements equation (A11) acquires the more familiar form

$$f(g) = \sum_{n=0}^{\infty} \int_{\rho=0}^{\infty} \left[\sum_{j_1,j_2=n}^{\infty} \sum_{q_1=-j_1}^{j_1} \sum_{q_2=-j_2}^{j_2} \bar{D}^{n,\rho}_{j_1 q_1 j_2 q_2}(g) f^{j_1 q_1 j_2 q_2}_{n,\rho} \right] (n^2+\rho^2) d\rho, \tag{A24}$$

where

$$f^{j_1 q_1 j_2 q_2}_{n,\rho} = \int f(g) D^{n,\rho}_{j_1 q_1 j_2 q_2}(g) dg, \tag{A25}$$

and the quantity in brackets represents the trace in (A11). In the same way we can translate equation (A16) obtaining

$$\delta(g) = \sum_{n=0}^{\infty} \int_{\rho=0}^{\infty} \left[\sum_{j=n}^{\infty} \sum_{q=-j}^{j} \bar{D}^{n,\rho}_{jqjq}(g) \right] (n^2+\rho^2) d\rho = \sum_{n=0}^{\infty} \int_{\rho=0}^{\infty} \mathrm{Tr}\left[\bar{D}^{n,\rho}(g) \right] (n^2+\rho^2) d\rho. \tag{A26}$$

Using equations (A22) and (A23), we can compute

$$\int_{SU(2)} D^{n,\rho}_{j_1 q_1 j_2 q_2}(u)\ du = \delta_{j j_2} \int_{SU(2)} D^{j}_{q q_2}(u) du = \delta_{j_2 0} \delta_{j_1 0}. \tag{A27}$$

b. On the tensor product of two irreducible representations

The tensor product of two irreducible representations of the principal series $T_{n_1\rho_1}$ and $T_{n_2\rho_2}$ decomposes into a direct integral of irreducible representations $T_{n\rho}$ for those n's such that $n+n_1+n_2$ is an even integer and no restriction for ρ. For a proof of this assertion, and for explicit realizations of the tensor product of two representations of the principal series see [29].

c. Generators and Casimir operators

An infinitesimal $g \in SL(2,C)$ in the adjoint representation can be parametrized by the six numbers $\lambda_{\mu\nu} = -\lambda_{\nu\mu}$ as

$$g = e + i\lambda_{\mu\nu} L^{\mu\nu} + O(\lambda^2), \tag{A28}$$

where $iL^{\mu\nu} \in sl(2,c)$, the algebra of $SL(2,C)$. The corresponding irreducible representation operator $T^{n\rho}(g)$ has the form

$$T_{n\rho}(g) = \hat{1} + i\lambda_{\mu\nu} \hat{L}^{\mu\nu}_{(n\rho)} + O(\lambda^2). \tag{A29}$$

There are two Casimir operators in $SL(2,C)$ corresponding to $L_{\mu\nu}L^{\mu\nu}$ and $L_{\mu\nu}{}^{*}L^{\mu\nu}$ respectively, namely

$$\hat{\mathcal{C}}_{1(n,\rho)} = \hat{L}_{(n\rho)\mu\nu} \hat{L}^{\mu\nu}_{(n\rho)} = \frac{1}{4}(n^2 - \rho^2 - 4)\ \hat{1}, \tag{A30}$$

and

$$\hat{\mathcal{C}}_{2(n,\rho)} = \epsilon_{\mu\nu\alpha\beta} \hat{L}^{\mu\nu}_{(n\rho)} \hat{L}^{\alpha\beta}_{(n\rho)} = \frac{1}{4} n\rho\ \hat{1}. \tag{A31}$$

[1] JW Barrett, L Crane, *A Lorentzian Signature Model for Quantum General Relativity*, gr-qc/9904025, Class Quant Grav 17 (2000) 3101-3118.

[2] A Perez, C Rovelli, *Spin foam model for Lorentzian General Relativity*, gr-qc/0009021.

[3] M Reisenberger, *Worldsheet formulations of gauge theories and gravity*, talk given at the 7th Marcel Grossmann Meeting Stanford, July 1994; gr-qc/9412035.

[4] J Iwasaki, *A definition of the Ponzano-Regge quantum gravity model in terms of surfaces*, J Math Phys 36 (1995) 6288.

[5] J Baez, *Spin Foam Models*, Class Quant Grav 15 (1998) 1827-1858; gr-qc/9709052. *An Introduction to Spin Foam Models of Quantum Gravity and BF Theory*, to appear in to appear in “Geometry and Quantum Physics”, eds Helmut Gausterer and Harald Grosse, Lecture Notes in Physics (Springer-Verlag, Berlin); gr-qc/9905087.

[6] M Reisenberger, C Rovelli, *Sum over Surfaces form of Loop Quantum Gravity*, Phys Rev D56 (1997) 3490-3508. C Rovelli, *Quantum gravity as a sum over surfaces*, Nucl Phys B57 (1997) 28-43. C Rovelli, *The projector on physical states in loop quantum gravity*, gr-qc/9806121.

[7] R De Pietri, *Canonical Loop Quantum Gravity and Spin Foam Models*, Proceeding of the XXIII SIGRAV conference, Monopoli (Italy), September 1998.

[8] L Freidel, K Krasnov, *Spin Foam Models and the Classical Action Principle*, Adv Theor Math Phys 2 (1999) 1183-1247, hep-th/9807092.

[9] JW Barrett, L Crane, *Relativistic spin networks and quantum gravity*, J Math Phys 39 (1998) 3296.

[10] C Misner, *Feynman quantization of General Relativity*, Rev Mod Phys 29 (1957) 497.

[11] SW Hawking, *The Path-Integral Approach to Quantum Gravity*, in “General Relativity: An Einstein Centenary Survey”, SW Hawking and W Israel eds (Cambridge University Press, Cambridge 1979).

[12] M Reisenberger, *A left-handed simplicial action for Euclidean general relativity*, Class Quantum Grav 14 (1997) 1730–1770; gr-qc/9609002; gr-qc/9711052; gr-qc/9903112

[13] J Iwasaki, *A surface theoretic model of quantum gravity*, gr-qc/9903112. J Iwasaki, *A lattice quantum gravity model with surface-like excitations in 4-dimensional spacetime*, gr-qc/0006088.

[14] A Perez, C Rovelli, *A spin foam model without bubble divergences*, gr-qc/0006107.

[15] J F Plebanski,*On the separation of Einsteinian substructures*, J Math Phys 12, (1977) 2511.

[16] R De Pietri, L Freidel, K Krasnov, C Rovelli, *Barrett-Crane model from a Boulatov-Ooguri field theory over a homogeneous space*, Nuclear Physics, to appear, hep-th/9907154.

[17] M Reisenberger, C Rovelli, *Spinfoam models as Feynman diagrams*, gr-qc/0002083. M Reisenberger, C Rovelli, *Spacetime as a Feynman diagram: the connection formulation*, gr-qc/0002095.

[18] V Turaev, *Quantum invariants of 3-manifolds and a glimpse of shadow topology* in *Quantum Groups*, Springer Lecture Notes in Mathematics 1510, pp 363-366 (Springer-Verlag, New York, 1992); “Quantum Invariants of Knots and 3-Manifolds” (de Gruyter, New York, 1994).

[19] H Ooguri, *Topological Lattice Models in Four Dimensions*, Mod Phys Lett A7 (1992) 2799.

[20] L Crane and D Yetter, *A Categorical construction of 4-D topological quantum field theories*, in “Quantum Topology”, L Kaufmann and R Baadhio eds (World Scientific, Singapore 1993); hep-th/9301062.

[21] L Crane, L Kauffman and D Yetter, *State-Sum Invariants of 4-Manifolds*, J Knot Theor Ramifications 6 (1997) 177–234; hep-th/9409167.

[22] C Rovelli, L Smolin, *Discreteness of the area and volume in quantum gravity*, Nucl Phys B442 (1995), 593-622. Erratum: Nucl Phys B456 (1995), 734.

[23] A Perez, Carlo Rovelli, *Finiteness of a spinfoam model for euclidean quantum general relativity*, very soon on gr-qc.

[24] J Baez, J W Barret, *Integrability for relativistic spin networks*, in preparation.

[25] JW Barrett, *The classical evaluation of relativistic spin networks*, Adv Theor Math Phys 2 (1998) 593-600.

[26] IM Gel'fand, MI Graev, N Ya Vilenkin, “Generalized Functions”, volume 5, Integral Geometry and Representation Theory, (Academic Press, New York 1966).

[27] W Ruhl, “The Lorentz Group and Harmonic Analysis” (WA Benjamin Inc, New York 1970).

[28] A Held, ET Newman, R Posadas, *The Lorentz Group and the Sphere*, J Math Phys 11 (1970) 3145.

[29] MA Naimark, *Decomposition of a tensor product of irreducible representations of the proper Lorentz group into irreducible representations.*, Amer Math Soc Translations ser 2 36 101-136 (1964).

[30] D Oriti, R M Williams, *Gluing 4-simplices: a derivation of the Barrett-Crane spin foam model for Euclidean quantum gravity*, gr-qc/0010031.